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Homogeneous Bianchi Type VI_0 Perfect Fluid Space-Times

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An algorithm is presented for generating new exact solutions of the Einstein equations for spatially homogeneous cosmological models of Bianchi type VI_0. The energy-momentum tensor is of perfect fluid type. Starting from Dunn and Tupper's dust-filled universe, new classes of solutions are obtained. The solutions represent anisotropic universes filled with perfect fluid not satisfying the equation of state. Some of their physical properties are studied.

1. INTRODUCTION

Experimental studies of the isotropy of cosmic microwave radiation and speculation about the amount of helium formed at early stages of the universe and many other effects have stimulated theoretical interest in anisotropic cosmological models. The spatially homogeneous Bianchi models I–IX necessarily admit a three-parameter group which acts simply transitively over the three-dimensional constant-time subspace. The importance of Bianchi spaces is due to the simplicity of the field equations. The relative ease of solution has made these spaces useful in constructing and studying models of spatially homogeneous cosmologies. Homogeneous cosmological models filled with matter together with specified equations of state have already been widely studied. I confine myself to the class of spatially homogeneous Bianchi type VI_0 spaces. The solutions of the Einstein equations in the case of stiff matter were obtained by Ellis and MacCallum (1969). Collins (1971) and Ruban (1978) presented exact solutions of type VI_0 with a perfect fluid. A class of cosmological models with a perfect fluid and electromagnetic field was also investigated by Dunn and Tupper (1976, 1987) and Tupper (1977) for the case of dust. Lorentz (1982) generalized

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the solution given by Ellis and MacCallum (1969). Roy and Singh (1983) derived some exact solutions of Einstein–Maxwell equations representing a free gravitational field of the magnetic type with matter and an incident magnetic field. Recently, Ribeiro and Sanyal (1987) studied spatially homogeneous Bianchi type VI_0 models containing a viscous fluid in the presence of an axial magnetic field.

Hajj-Boutros (1984) presented a technique for generating exact solutions of the Einstein equations with spherical symmetry. He also applied the technique to build exact solutions in the case of hypersurface-homogeneous space-times and Bianchi type II spaces (Hajj-Boutros, 1985, 1986). Following his method, I derive here an algorithm to generate exact solutions of the Einstein equations for the Bianchi type VI_0 class of models with energy-momentum tensor of a perfect fluid. Starting from Dunn and Tupper's (1976) dust-filled universe, I obtain two new classes of solutions which can be added to rare perfect fluid solutions not satisfying the equation of state, e.g., solutions obtained by Singh and Singh (1968), Singh and Abdussattar (1973, 1974), Patel and Vaidya (1969), and Shri Ram (1988). I also study the geometrical and physical properties of the solutions obtained.

2. FIELD EQUATIONS AND GENERATION TECHNIQUES

The metric for the Bianchi type VI_0 class of models is taken of the form

$$ds^2 = -dt^2 + A(t)\,dx^2 + B(t)\,e^{-x}\,dy^2 + C(t)\,e^{x}\,dz^2 \tag{1}$$

where A, B, and C are cosmic scale functions. The field equations in general relativity are

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \tag{2}$$

In the case of the energy-momentum tensor of a perfect fluid

$$T_{\mu\nu} = (\rho + p)v_\mu v_\nu + pg_{\mu\nu} \tag{3}$$

where v^μ is the 4-velocity vector, p is the pressure and ρ is the mass-energy density.

For the metric (1) the field equations to be considered are

$$\tfrac{1}{2}(B''/B) + \tfrac{1}{2}(C''/C) - \tfrac{1}{4}(B'/B)^2 - \tfrac{1}{4}(C'/C)^2 + \tfrac{1}{4}(B'C'/BC) + 1/(4A) = -p \tag{4}$$

$$\tfrac{1}{2}(A''/A) + \tfrac{1}{2}(C''/C) - \tfrac{1}{4}(A'/A)^2 - \tfrac{1}{4}(C'/C)^2 + \tfrac{1}{4}(A'C'/AC) - 1/(4A) = -p \tag{5}$$

$$\tfrac{1}{2}(A''/A) + \tfrac{1}{2}(B''/B) - \tfrac{1}{4}(A'/A)^2 - \tfrac{1}{4}(B'/B)^2 + \tfrac{1}{4}(A'B'/AB) - 1/(4A) = -p \tag{6}$$

$$\tfrac{1}{4}(A'B'/AB)+\tfrac{1}{4}(A'C'/AC)+\tfrac{1}{4}(B'C'/BC)-1/(4A)=\rho \qquad (7)$$

$$(B'/B)-(C'/C')=0 \qquad (8)$$

where a prime denotes differentiation with respect to t.

From equation (8) we get

$$B=nC \qquad (9)$$

where n is a constant. Without loss of generality we can take $n=1$. Elimination of p from (4) and (5) gives the condition of isotropy of pressures

$$2(B''/B-A''/A)+(A'/A)(A'/A-B'/B)+2/A=0 \qquad (10)$$

To treat equation (10), I introduce functions R and S defined by

$$R=A'/A, \qquad S=B'/B \qquad (11)$$

Using (11), equation (10) becomes

$$S'+S^2-R'-\tfrac{1}{2}R^2-\tfrac{1}{2}RS+1/A=0 \qquad (12)$$

which can be regarded a Riccati equation in S (or R).

Case 1. If (12) is regarded a Riccati equation in S, we linearize it by the change of function

$$S=S_0+1/Z \qquad (13)$$

where S_0 is a particular solution of (12) with S being the more general one. From equations (12) and (13), we obtain

$$Z'+Z(\tfrac{1}{2}R-2S_0)=1 \qquad (14)$$

Equation (14) has the general solution

$$Z=(B_0^2/A^{1/2})\left[\int (A^{1/2}/B_0^2)\,dt+k_1\right] \qquad (15)$$

k_1 is an integration constant. From (13) and (15) we obtain

$$B=B_0\exp\left\{\int \frac{dt}{(B_0^2/A^{1/2})[\int (A^{1/2}/B_0^2)\,dt+k_1]}+k_2\right\} \qquad (16)$$

where k_2 is another constant of integration. Hence, from the couple $[A, B_0]$, this algorithm allows us to obtain $[A, B]$, where B is given by (16) and A stays invariable.

Case 2. When (12) is regarded a Riccati equation in R, it can be linearized by the change of function

$$R=R_0+1/Y \qquad (17)$$

Using (17) in equation (12), we get

$$Y' + Y(-R_0 - \tfrac{1}{2}S) = \tfrac{1}{2} \tag{18}$$

where R_0 is a particular solution of (12) with R being the more general one. The general solution of (18) is

$$Y = (A_0 B^{1/2})\left[\int (1/2A_0B^{1/2})\, dt + k_3\right], \tag{19}$$

k_3 is a constant. Equations (17) and (19) yield

$$A = A_0 \exp\left\{\int \frac{dt}{(A_0B^{1/2})[\int (1/2A_0B^{1/2})\, dt + k_3]} + k_4\right\} \tag{20}$$

where k_4 is an integration constant. Thus, from the metric functions $[A_0, B]$ we can generate new functions $[A, B]$, where A is given by (20) and B stays invariable.

3. GENERATED SOLUTIONS

I confine myself to the Bianchi type VI_0, spatially homogeneous, dust-filled cosmology with metric (Dunn and Tupper, 1976)

$$ds^2 = -dt^2 + t^2\, dx^2 + t(e^{-x}\, dy^2 + e^x\, dz^2) \tag{21}$$

The mass-energy density of this model is

$$\rho = 1/t^2 \tag{22}$$

To apply the formula (16), take

$$A = t^2, \qquad B_0 = t \tag{23}$$

Inserting the values of A and B into (16), we obtain

$$A = t^2, \qquad B = t(b \log at) \tag{24}$$

where a and b are arbitrary constants. By change of scale the metric of the new solution (24) can be written as

$$ds^2 = -dt^2 + t^2\, dx^2 + (t \log at)(e^{-x}\, dy^2 + e^x\, dz^2) \tag{25}$$

Starting from the metric (25) as a particular solution, the formula (16) provides

$$A = t^2, \qquad B = mt(b \log at + 1) \tag{26}$$

where m and b are arbitrary constants. Thus, the metric of the new class of solutions (26) is

$$ds^2 = -dt^2 + t^2\, dx^2 + t(b \log at + 1)(e^{-x}\, dy^2 + e^x\, dz^2) \tag{27}$$

Applying the formula (16) for the metric (27) as a particular solution, one arrives at (27) again with different parameters. I call (27) the M_1 class of models. For $b=0$ one obtains the Dunn and Tupper (1976) solution (21).

If one applies the formula (20) for the metric (21) as a particular solution, one obtains

$$A=(\alpha t+\beta t^{-1/2})^2, \qquad B=t \tag{28}$$

α, β are arbitrary constants. The metric of the new class of solutions reads

$$ds^2=-dt^2+(\alpha t+\beta t^{-1/2})^2\, dx^2+t(e^{-x}\, dy^2+e^x\, dz^2) \tag{29}$$

I call this class M_2. For $\alpha=1$ and $\beta=0$, one arrives at the metric (21).

4. PHYSICAL PROPERTIES OF SOLUTIONS

The pressure and energy-density are given by (4) and (7). For the M_1 class of models, we obtain

$$p=b^2/4t^2(b\log at+1)^2-b/2t^2(b\log at+1) \tag{30}$$

$$=3b/2t^2(b\log at+1)+b^2/4t^2(b\log at+1)^2+1/t^2 \tag{31}$$

The energy-density ρ is positive for $b>0$. The dominant energy conditions of Hawking and Ellis (1973), i.e.,

$$\rho>0, \qquad \rho+3p>0 \tag{32}$$

are identically satisfied. The pressure and energy-density do not satisfy the equation of state of the form

$$p=(\gamma-1)\rho, \qquad 1\le\gamma\le 2 \tag{33}$$

and thus the solutions M_1 belong to some rare perfect fluid solutions of this type existing in literature.

Equations (30) and (31) show that the pressure and energy density are infinite for $t=0$. Also, as $t\to 0$ the proper volume $V=AB^2\to 0$. Thus, $t=0$ represents a pointlike singularity. As $t\to\infty$, $p(\infty)=\rho(\infty)=0$ and thus this class of models gives essentially empty universes for large t.

I now discuss the kinematic behavior of the class M_1. The projection tensor

$$h_{\mu\nu}=g_{\mu\nu}+v_\mu v_\nu \tag{34}$$

is used for splitting the covariant derivative of the velocity vector v^μ as follows:

$$v_{\mu;\nu}=-\dot{v}_\mu v_\nu+\omega_{\mu\nu}+\sigma_{\mu\nu}+\tfrac{1}{3}\theta h_{\mu\nu} \tag{35}$$

where $\dot{v}_\mu$, θ, $\omega_{\mu\nu}$, and $\sigma_{\mu\nu}$ are acceleration, scalar expansion, rotation tensor, and shear tensor, respectively. The shear tensor plays an important role in general relativistic cosmological and stellar models (Collins and Wainwright, 1983). For this class all fluids are acceleration and rotation-free, but they do have expansion θ and shear scalar σ given by

$$\theta = (b+2+2b\log at)/t(b\log at+1) \tag{36}$$

$$\sigma = (1/2\sqrt{3})(b\log at+1-b)/t(b\log at+1) \tag{37}$$

Equation (36) shows that the expansion is (1) infinite for $t=0$, (2) nonzero for all t $(0<t<\infty)$, and (3) zero as $t\to\infty$. Thus, M_1 represents a class of expanding universes. Equation (37) shows that (1) the scalar shear is nonzero for all values of t, $0<t<\infty$, and thus, the model is anisotropic, (2) at infinite time $(t\to\infty)$ the model is shear-free, and thus, there is no anisotropy. We also find that

$$\sigma/\theta \to 1/4\sqrt{3} \qquad \text{as} \quad t\to\infty \tag{38}$$

which shows that the shear scalar does not tend to zero faster than the expansion.

The distribution of matter and kinematical parameters for the class M_2 are

$$p = 1/4t^2 - 1/4(\alpha t+\beta t^{-1/2})^2 \tag{39}$$

$$\rho = 1/4t^2 - 1/4(\alpha t+\beta t^{-1/2})^2 + (\alpha - \tfrac{1}{2}\beta t^{-3/2})/(\alpha t+\beta t^{-1/2}) \tag{40}$$

$$\theta = 1/t + (\alpha - \tfrac{1}{2}\beta t^{-3/2})/(\alpha t+\beta t^{-1/2}) \tag{41}$$

$$\sigma = (1/4\sqrt{3})(2\alpha t - 3\beta t^{-1/2})/t(\alpha t+\beta t^{-1/2}) \tag{42}$$

The dominant energy conditions are satisfied if $\alpha>1$ and $\beta>0$. The other kinematical quantities, rotation and acceleration, are zero. The physical behavior of the class M_2 is similar to the class M_1.

In conclusion, I have presented spatially homogeneous and anisotropic cosmological models of Bianchi type VI_0 filled with perfect fluid. The models are expanding and shearing, which give essentially empty universes for large time. The behavior of the fluid is time dependent and can be physically reasonable.

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